Effect of the Hyperfine Splitting of a μ-Mesonic Atom on Its Lifetime

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The lifetime of a μ-mesonic atom depends on its hyperfine state because of the spin dependence of the interaction between the μ^- meson and the capturing proton. The small lifetime difference between the hyperfine states is estimated in a one-particle model. Possible experimental detection of such a difference is discussed.

THE purpose of this paper is to point out that the lifetime of a μ-mesonic atom is affected by its hyperfine splitting, and that a measurement of the difference of the lifetimes in different hyperfine states may yield useful information on the capture interaction between the μ meson and the nucleons.

I

We consider a μ^- meson in the K orbit around a nucleus of spin I (where $I>0$) and atomic number Z. The hyperfine splitting of the system corresponds to a frequency which is in all cases much bigger than the inverse of the lifetime of the μ-mesonic atom. The system therefore exists in two incoherent hyperfine states corresponding to the total angular momenta $F=I+\frac{1}{2}$ and $F=I-\frac{1}{2}$. For these two states the probabilities per second against the distintegration $\mu^-\rightarrow e^-+\nu+\bar{\nu}$ is clearly the same (except for the negligibly small perturbation on the electron wave function due to the nuclear magnetic moment). The capture probabilities, however, are in general not the same because of the combined action of the following three effects: (i) In the two hyperfine states the correlation between the spin of μ^- and $\mathbf{I}$ is different. (ii) There is in general a correlation between the spin of the proton in the nucleus and $\mathbf{I}$. This is especially true for nuclei with odd Z and odd A. (iii) The capture probability of a μ^- by a proton depends on their relative spin orientation.

We make here a rough estimate of the difference of the capture probabilities per second: λ_+ and λ_-, for the two hyperfine states $F=I+\frac{1}{2}$ and $F=I-\frac{1}{2}$ for a nucleus with odd Z and odd A. The nucleus is taken to consist of one "outside" proton and a core with even numbers of protons and neutrons. The core is assumed to be spinless. Captures of the μ^- by the core protons are then independent of the orientation of I, and therefore contribute equally to λ_+ and λ_-. For the capture by the outside proton we make the approximation that it be regarded as free. The calculation of the difference between λ_+ and λ_- is then clearly separated into the estimation of the three effects enumerated in the last paragraph.

The magnitude of effect (iii) is easily computed: the capture probability of a μ^- by a free proton is proportional to

$$\langle|a+b\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\mu|\rangle, \qquad (1)$$

where the matrix element is taken over the spin wave functions of the μ^- and the proton in the initial state. In this expression,

$$a=|C_{\mathrm{F}}|^2+3|C_{\mathrm{GT}}|^2+|G_P|^2-(G_P^*C_{\mathrm{GT}}+\text{complex conjugate}), \qquad (2)$$
$$b=-2|C_{\mathrm{GT}}|^2+(C_{\mathrm{F}}^*C_{\mathrm{GT}}-\tfrac{1}{3}C_{\mathrm{F}}^*G_P+\tfrac{2}{3}G_P^*C_{\mathrm{GT}}+\text{complex conjugate}).$$

To obtain (1) and (2), we start from the relativistic interaction Hamiltonian

$$H=\textstyle\sum_i C_i(\psi_n^\dagger O_i\psi_p)(\psi_\nu^\dagger O_i\psi_\mu), \qquad (3)$$

where O_i has been previously defined.[1] We take the neutrino to be a left-handed, 2-component field. The nucleons are then taken to be nonrelativistic, for which case the interaction is characterized by the constants

$$C_{\mathrm{F}}=C_S+C_V,$$
$$C_{\mathrm{GT}}=C_T+C_A, \qquad (4)$$
$$G_P=C_P(2M)^{-1}q,$$

where q is the momentum of the neutrino and M the mass of the nucleon. (We take $c=\hbar=1$.)

In Eq. (1) the term $b\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\mu$ expresses the spin dependence of the capture rate. The value of $\langle|\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\mu|\rangle$ is easily calculated as a product of two factors, one expressing the projection of $\boldsymbol{\sigma}_\mu$ on I [due to effect (i)], and the other expressing the project of $\boldsymbol{\sigma}_p$ on I [due to effect (ii)]. The result is, for $I>0$:

$$\langle|\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\mu|\rangle=I^{-1}(I+1)^{-1}\times\{F(F+1)-I(I+1)-\tfrac{3}{4}\}\langle|\boldsymbol{\sigma}_p\cdot\mathbf{I}|\rangle, \qquad (5)$$

where

$$\langle|\boldsymbol{\sigma}_p\cdot\mathbf{I}|\rangle=I(I+1)-L(L+1)+\tfrac{3}{4}. \qquad (6)$$

In the last equation L is the "orbital angular momentum" of the outside proton in the nucleus. The difference $\lambda_+-\lambda_-$ is thus proportional to

$$b\{I(I+1)-L(L+1)+\tfrac{3}{4}\}(2I+1)I^{-1}(I+1)^{-1}. \qquad (7)$$

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[1] We use the notation O_i defined in Eq. (11) of T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957).

Now the average,

$$\bar{\lambda}_{\text{cap}}=(4I+2)^{-1}[(2I+2)\lambda_{+}+2I\lambda_{-}], \tag{8}$$

of λ_+ and λ_- consists of a contribution from the core and a contribution from the outside proton. The latter is proportional to a and the former to $(Z-1)a\xi$. The factor ξ here represents the ratio of the effect of the exclusion principle suppression on the final state, for capture by a core proton and by the outside proton. One therefore obtains

$$\frac{\lambda_+-\lambda_-}{\bar{\lambda}_{\text{cap}}}=\frac{b}{aZ'}\frac{2I+1}{I} \quad \text{for} \quad I=L+\tfrac{1}{2} \tag{9}$$

$$\frac{\lambda_+-\lambda_-}{\bar{\lambda}_{\text{cap}}}=\frac{-b}{aZ'}\frac{2I+1}{I+1} \quad \text{for} \quad I=L-\tfrac{1}{2}, \tag{10}$$

where

$$Z'=(Z-1)\xi+1. \tag{11}$$

II

Before discussing possible experimental measurements of the difference $\lambda_+-\lambda_-$, we make a few remarks on the approximations that led to (9) and (10):

(a) The quantity L in (9) and (10) has meaning strictly only in the one-particle model. A measure of the deviation from such a model is the deviation of the magnetic moment of the nucleus from the Schmidt line.[2] Depending on the nature of the configurations that have to be mixed with the single-particle states, different kinds of correction terms have to be added to (9) and (10). As a zeroth approximation, however, the Schmidt plot in general decides unambiguously whether the dominant single-particle state has $I=L+\frac{1}{2}$ or $I=L-\frac{1}{2}$, and consequently decides whether (9) or (10) obtains. If the difference $\lambda_+-\lambda_-$ is measured for a number of nuclei, the variation of its sign and its deviation from (9) and (10) may yield interesting information on the mixing of configurations for the shell model.

(b) Equations (1) and (2) were derived under the assumption that the proton be considered as free and at rest. To account for the effect of a momentum distribution of the proton, one only has to replace the neutrino momentum q and its square q^2, in the definition for G_P, by their respective averages.

Another question that we have briefly examined has a bearing on the validity of the approximation of taking the proton as free. We examined first those cases in which the neutrino is emitted with a wavelength that is not shorter than the radius of the nucleus. One can then approach the problem by the same method that one uses in the K capture of e^-. For long neutrino wavelengths in both e^- and μ^- capture, the neutrino carries away a total angular momentum $\frac{1}{2}$. We denote by I' the resultant angular momentum of *all* the nucleons after the capture of the μ^-. The origin of the difference $\lambda_+-\lambda_-$ lies, then, in the obvious fact that for $F=I+\frac{1}{2}$, the capture cannot proceed with $(I'-I)=-1$, and for $F=I-\frac{1}{2}$ it cannot proceed with $(I'-I)=+1$. A statistical argument about the distribution of the final states then leads to the same results as (9) and (10).

For neutrino wavelengths that are not necessarily long compared to the size of the nucleus, a modified closure method leads also to essentially the same results as (9) and (10).

[2] See, e.g., M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955), or E. Feenberg, *Shell Theory of the Nucleus* (Princeton University Press, 1955).

III

The fractional difference given by (9) or (10) is, in the neighborhood of $Z=10$, of the order of 20% or perhaps less. (This difference is especially large if $C_F\cong-C_{GT}$.) For $Z\sim10$ it is experimentally known that

$$\bar{\lambda}_{\text{cap}}\approx\lambda_{\text{decay}}\approx\tfrac{1}{2}\lambda_{\text{total}},$$

where λ_{decay} denotes the natural decay probability of μ^- per second, and λ_{total} the sum of capture probability and decay probability. The fractional difference in λ_{total} or the lifetimes for these two hyperfine states may therefore be ~10 percent. To observe such differences it seems best to measure the number of *decay electrons* from the μ^- as a function of time and demonstrate that it is not a simple exponential. We discuss below a few points connected with such measurements.

(a) To measure a small fractional lifetime difference $\delta(\lesssim10\%)$ with a finite number N of counts of decay electrons from the mesonic atoms, it is best to measure the curvature of the logarithm of the decay curve against time. It turns out that the best time region to measure such curvature extends from zero time to a few (~4) lifetimes. It is not difficult to show that what one measures this way is the quantity δ^2 to an accuracy that is of the order of $\pm e^2N^{-\frac{1}{2}}$. To establish the existence of a δ^2 to outside of M standard deviations, one therefore must have

$$\delta^2>Me^2N^{-\frac{1}{2}},$$

i.e.,

$$N>M^2e^4\delta^{-4}. \tag{12}$$

For $M=3$, $\delta=10\%$, one therefore needs more than 5×10^6 decay electron counts, a number that is within the possibilities of existing cyclotrons.

(b) In order that no spurious effects be introduced, one must take care that the μ^- mesons be captured in a substance that is rather pure chemically. Since contaminations of long lifetimes introduce large errors, one must minimize the amount of impurities consisting of light atoms in the stopping material. E.g., the presence of a fraction α of counts from a contamination with twice the lifetime of the μ-mesonic atom in question introduces, it can be shown, an error (absolute, not relative) in the determination of δ^2 of the order of 3α.

It is therefore necessary that such contaminations be less than $0.1\delta^2$ (which is 0.1% for $\delta \approx 0.1$) for a 30% accuracy in the determination of δ^2.

Since different isotopes capture μ mesons with approximately the same rate, contamination of the stopping substance with its isotopes can be shown to produce relatively small curvature in the logarithmic decay curve, and consequently can be well tolerated. An isotopic contamination of a few percent will be quite harmless.

(c) A curvature measurement, however, does not allow for a determination of the sign of δ, even if one knows the population of the two hyperfine states. A measurement of the change of curvature, or a study of the time dependence of the angular asymmetry of the decay electrons if the μ meson is not completely depolarized in slowing down, is necessary to determine the sign of δ. Both of these seem to be very difficult.

(d) For a nucleus with an even number of protons the difference $\lambda_+ - \lambda_-$, if it exists, should be very small. Also, if $I=0$, there should be only one lifetime. These obvious conclusions offer convenient "controls" in any experimental setup to detect $\lambda_+ - \lambda_-$.

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Conservation Laws in General Relativity

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The conservation laws are examined from the transformation properties of the Lagrangian. The energy-momentum complex obtained has mixed indices, $T_\mu{}^\nu$, whereas a symmetric quantity $\mathcal{T}^{\mu\nu}$ is required for the definition of angular momentum. Such a symmetric quantity has been constructed by Landau and Lifshitz. In the course of examining the relationship between these quantities, two hierarchies of complexes $T_{(n)\mu}{}^\nu$ and $\mathcal{T}_{(n)}{}^{\mu\nu}$ are constructed. Under linear coordinate transformations the former are tensor densities of weight $(n+1)$ and the latter of weight $(n+2)$. For $n=0$ these reduce to the canonical $T_\mu{}^\nu$ and the Landau-Lifshitz $\mathcal{T}^{\mu\nu}$, respectively.

By requiring the energy-momentum complex to generate the coordinate transformations, and the total energy and momentum to form a free vector, one can identify the canonical complex $T_\mu{}^\nu$ as the appropriate quantity to describe the energy and momentum of the field plus matter. Similarly, by requiring the total angular momentum to behave as a free antisymmetric tensor, one can construct, in the usual manner, an appropriate quantity from $\mathcal{T}_{(-1)}{}^{\mu\nu}$. The angular momentum complex so defined differs from that proposed by Landau and Lifshitz as well as from an independent construction by Bergmann and Thomson.

1. INTRODUCTION

CONSERVATION laws in general relativity were first formulated by Einstein in 1916.[1] By examining the behavior of the Lagrangian of the theory of gravitation under infinitesimal translations of the coordinate system, he was led to the canonical energy-momentum pseudotensor of the gravitational field. Because of the nontensor character of the pseudotensor, the local energy density of the field does not have a covariant significance. Indeed, Schrödinger[2] criticized this formulation of the conservation laws because he found a coordinate system in which all components of the pseudotensor vanished for the Schwarzschild metric. This criticism was answered only when Einstein[3] showed that total energy and momentum, the only physically meaningful quantities, are constants of the motion and transform as a free-vector[4] under linear coordinate transformations.

Except for a further examination of the relationship between conservation laws and transformation properties,[5] little has been added to the analysis by Einstein. However, in order to discuss angular momentum, a symmetric quantity for energy-momentum is desirable,[6] although not necessary.[7] The canonical pseudotensor has mixed indices, and raising one with the metric tensor does not yield a symmetric quantity. Recently

[1] A. Einstein, Berlin Ber. **42**, 1111 (1916).
[2] E. Schrödinger, Physik Z. **19**, 4 (1918).
[3] A. Einstein, Berlin Ber. 448 (1918); W. Pauli, *Relativitatstheorie* (B. G. Teubner, Leipzig, 1922), Enzyklopädie der Mathematische Wissenschaften, Vol. 2, p. 740.

[4] A free-vector is a set of quantities which are not defined at a particular point in space, yet which transform together as a vector under linear coordinate transformations.
[5] P. G. Bergmann, Phys. Rev. **75**, 680 (1949); P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).
[6] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).
[7] P. G. Bergmann and R. Thomson, Phys. Rev. **89**, 400 (1953).