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FIBRE BUNDLES AND THE PHYSICS OF THE MAGNETIC MONOPOLE

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It is a privilege on my part to speak at this symposium honoring Professor Chern. I think I can claim, among all the participants here, to have known him for the longest time, perhaps even longer than Mrs. Chern has known him.

I shall first discuss some recent developments in the theory of the magnetic monopole, using ideas borrowed from fiber bundle theory. I shall then make some remarks about the relationship between mathematics and physics.

1. Magnetic Monopole and Non-trivial Bundles

The magnetic monopole is the magnetic charge. While the idea of magnetic monopoles must have been discussed in classical physics early in the history of electricity and magnetism, modern discussions date back to 1931 in the important paper of Dirac[1] in which he pointed out that magnetic monopoles in quantum mechanics exhibit some extra and subtle features. In particular, with the existence of a magnetic monopole of strength g, electric charges and magnetic charges must necessarily be quantized, in quantum mechanics. We shall give a new derivation of this result in a few minutes.

If one wants to describe the wave function of an electron in the field of a magnetic monopole, it is necessary to find the vector potential $\vec{A}$ around the monopole. Dirac chose a vector potential which has a string of singularities. The necessity of such a string of singularities is obvious if we prove the following theorem[2].

Theorem: Consider a magnetic monopole of strength $g \neq 0$ at the origin and consider a sphere of radius R around the origin. There does not exist a vector potential $\vec{A}$ for the monopole magnetic field which is singularity free on the sphere.

This theorem can be easily proved in the following way. If there were a singularity free $\vec{A}$ we consider the loop integral

$$\oint A_\mu dx^\mu$$

around a parallel on the sphere as indicated in Figure 1. By Stoke's theorem this loop integral is equal to the total magnetic flux through the cap alpha:

$$\oint A_\mu dx^\mu = \Omega_\alpha \, . \tag{1}$$

Similarly we can apply Stoke's theorem to cap β obtaining

$$\oint A_\mu dx^\mu = \Omega_\beta \, . \tag{2}$$

Here Ω_α and Ω_β are the total upward magnetic flux through the caps α and β, both of which are bordered by the parallel. Subtracting these two equations we obtain

$$0 = \Omega_\alpha - \Omega_\beta \tag{3}$$

which is equal to the total flux out of the sphere, which in turn is equal to $4\pi g \neq 0$. We have thus reached a contradiction.

Having proved this theorem, we observe that R is arbitrary. Thus one concludes that there must be a string of singularities or strings of singularities in the vector potential to describe the monopole field. Yet we know that the magnetic field around the monopole is singularity free. This suggests that the string of singularities is not a real physical difficulty.

The problem here is of course well known to the mathematicians. The string of singularity is an "obstruction". It can be removed if we allow a piecing together of different vector potentials $\vec{A}$ in different regions of space. In other words, we use the idea used for the description of a manifold.

To be more specific, divide space outside the monopole into two overlapping regions. Call the region above the lower cone in Figure 2, region R_a, and the region below the upper cone, region R_b. The union of the two gives all points outside of the origin where the monopole is situated. In R_a we shall choose a vector potential for which there is only one non-vanishing component of A, the azimuthal component:

$$(A_r)^a = (A_\theta)^a = 0 \ , \quad (A_\varphi)^a = \frac{g}{r\sin\theta}(1-\cos\theta). \tag{4}$$

It is important to notice that this vector potential has no singularities anywhere in R_a. Similarly in R_b we choose the vector potential

$$(A_r)^b = (A_\theta)^b = 0 \ , \quad (A_\phi)^b = \frac{-g}{r\sin\theta}(1+\cos\theta) \tag{5}$$

which has no singularities in R_b. It is simple to prove that the curl of either of these two potentials gives correctly the magnetic field of the monopole.

In the region of overlap, since both of the two sets of vector potentials share the same curl, the difference between them must be curl-less and therefore must be a gradiant. Indeed a simple calculation shows

$$(A_\mu)^a - (A_\mu)^b = \partial_\mu \alpha, \text{ where } \alpha = 2g\phi, \tag{6}$$

where ϕ is the azimuthal angle. The schrodinger equation for an electron in the monopole field is thus

$$\left[\frac{1}{2m}(p-eA_a)^2+V\right]\psi_a = E\psi_a \quad \text{in } R_a,$$
$$\left[\frac{1}{2m}(p-eA_b)^2+V\right]\psi_b = E\psi_b \quad \text{in } R_b,$$

where ψ_a and ψ_b are respectively the wave functions in the two regions. The fact that the two vector potentials in these two equations are different by a gradient tells us, by the well-known gauge principle, that ψ_a and ψ_b are related by a phase factor transformation

$$\psi_a = S\psi_b \qquad S = e^{ie\alpha}, \tag{7}$$

or

$$\psi_a = e^{2iq\phi}\psi_b \ , \ q = eg. \tag{8}$$

Around the equator which is entirely in R_a, ψ_a is single valued. Similarly, since the equator is also entirely in R_b, ψ_b is single valued around the equator. Therefore, S must return to its original value when one goes around the equator. That implies Dirac's quantization condition:

$$2q = 2eg = \text{integer}. \tag{9}$$

Two ψ's, ψ_a and ψ_b, in R_a, and R_b respectively, that satisfy the condition of transition (8) in the overlap region, are called a section by the mathematicians. We see that around a monopole the electron wave function is a section and not an ordinary function. We shall call these wave sections. Different wave sections (belonging to different energies, for example), clearly satisfy the same condition of transition (8) with the same q. Thus we need to develop[3] the concept of a Hilbert space of sections. This can be done in a natural fashion and the physics of the magnetic monopole based on these ideas has been [3,4] developed.

The section defined by (8) is a section on a non-trivial bundle. The quantization condition, (9) above, already given[1] in 1931 by Dirac, is the simplest case of the deep Chern-Weil theorem[5] applied to a U_1 bundle over S_2.

The application of the concept of non-trivial bundles to the theory of magnetic monopoles thus removes the string singularity which has impeded progress in this field ever since Dirac's paper.

2. Mathematics and Physics

The development of physics in the twentieth century is characterized by the repeated borrowing from mathematics at the fundamental conceptual level:

Physics	Mathematics
Special Relativity	4-dimensional Space-time
General Relativity	Riemannian Geometry
Quantum Mechanics	Hilbert Space
Electromagnetism and Non-Abelian Gauge Fields	Fiber Bundles

Yet, it should be emphasized, that in each of these cases, the conceptual origin of the physical development was rooted in physics, and not in mathematics. There was, in fact, often a certain amount of resistance among physicists to the mathematization of physics. That this is the case is not surprising, since the fundamental value judgement in physics resides in relevance with respect to the physical universe around us. Physicists are therefore by training more pragmatic, so to say, than mathematicians if that is the right way to describe it. A good example of the resistance to the mathematization of physics can be found in a letter Faraday wrote to Maxwell in 1857. Faraday was a great experimental physicist with deep intuition, but not much mathematical training. He was the one who originated the concept of the lines of force. Maxwell, forty years Faraday's junior, set about to express Faraday's ideas in mathematical language. Faraday was suspicious of such efforts, as a true experimentalist should be, since from the viewpoint of physics, most mathematical formalisms do not lead to real understanding, but at best, to pointless decorations, and at worst, to clutterings that impede progress. Faraday's attitude was vividly revealed in the following passage in his letter: (Faraday to Maxwell, March 25, 1857)

> ".....My Dear Sir - I received your paper, and thank you very much for it. I do not say I venture to thank you for what you have said about "Lines of Force," because I know you have done it for the interests of philosophical truth; but you must suppose it is work grateful to me, and gives me much encouragement to think on. I was at first almost frightened when I saw such mathematical force made to bear upon the subject, and then wondered to see that the subject stood it so well."

Faraday's reaction is all the more understandable if we take a look at the sketches he made of his experiments (Fig.3). It is with such mundane instruments that he grappled with the secrets of nature. No wonder he had little use for a discipline apparently far removed from reality.

Maxwell equations turned out to be the greatest achievement of the physics of the nineteenth century. In the hundred years since Maxwell, physicists have found that his equations give amazingly accurate and complete descriptions of much of the physical world. And the study of Maxwell equations played essential roles in the conceptual development of all the great revolutions in physics in the early twentieth century: special relativity, general relativity and quantum mechanics. It continues to do so now.

But why did nature choose Maxwell equations and not, for example, scalar equations? Today we know the answer to this question, though the full meaning and full implication of the answer remain to be worked out. Maxwell equations, indeed the equations describing <u>all</u> fundamental forces of nature are gauge field equations which are based on the geometrical concept of connections on fiber bundles.

The beauty and profound nature of the geometry of fiber bundles was to a large extent brought forth by the work[5] of the man we are here to honor today. I must admit, however, that the appreciation of this beauty came to physicists only in recent years. Let me use myself as an example. I was impressed as a graduate student in the forties by the gauge principle for electromagnetism, since it was, besides general relativity, the only principle known for choosing interactions. In 1954 Mills and I generalized this principle to isotopic spin symmetry. In so doing we were interested in the equations[6] and not the geometrical meaning of the equations. Around 1968 I realized that gauge fields, Non-Abelian as well as Abelian ones, can be formulated in terms of non-integrable phase factors, i.e. path dependent group elements. I asked my colleague Jim Simons about the mathematical meaning of these non-integrable phase factors and he told me they are related to connections on fiber bundles. But I did not then appreciate that fiber bundle was a deep mathematical concept. In 1975 I invited Jim Simons to give to the theoretical physicists at Stony Brook a series of lectures on differential forms and fiber bundles. I am grateful to him that he accepted the invitation and I was among the beneficiaries. Through these lectures T.T. Wu and I finally understood the concept of non-trivial bundles and the Chern-Weil theorem, and realized how beautiful and general the theorem is. We were thrilled to appreciate that the non-trivial bundle was exactly the concept with which to remove, in monopole theory, the string difficulty which had been bothersome for over forty years.

Later in 1975, Belavin, Polyakov, Schwartz and Tyupkin[7] found the instanton solution which in mathematical language is a connection on a non-trivial SU_2 bundle over S_4. It became clear that non-trivial bundles will play an important part in elementary particle theory. Powerful mathematics[8] using the Atiyah-Singer theorem and the methods of algebraic geometry have been brought to bear on the instanton problem.

It would be wrong, however, to think that the disciplines of mathematics and physics overlap that much. They do not (Fig. 4). And they have their separate aims and tastes. They have distinctly different value judgements, and they have different traditions. At the fundamental conceptual level they amazingly share some concepts, but even there, the life force of each discipline runs along its own veins.

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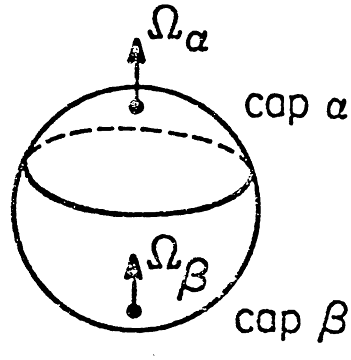

Fig. 1

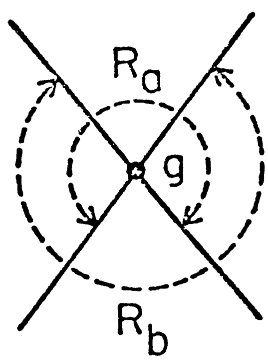

Fig. 2

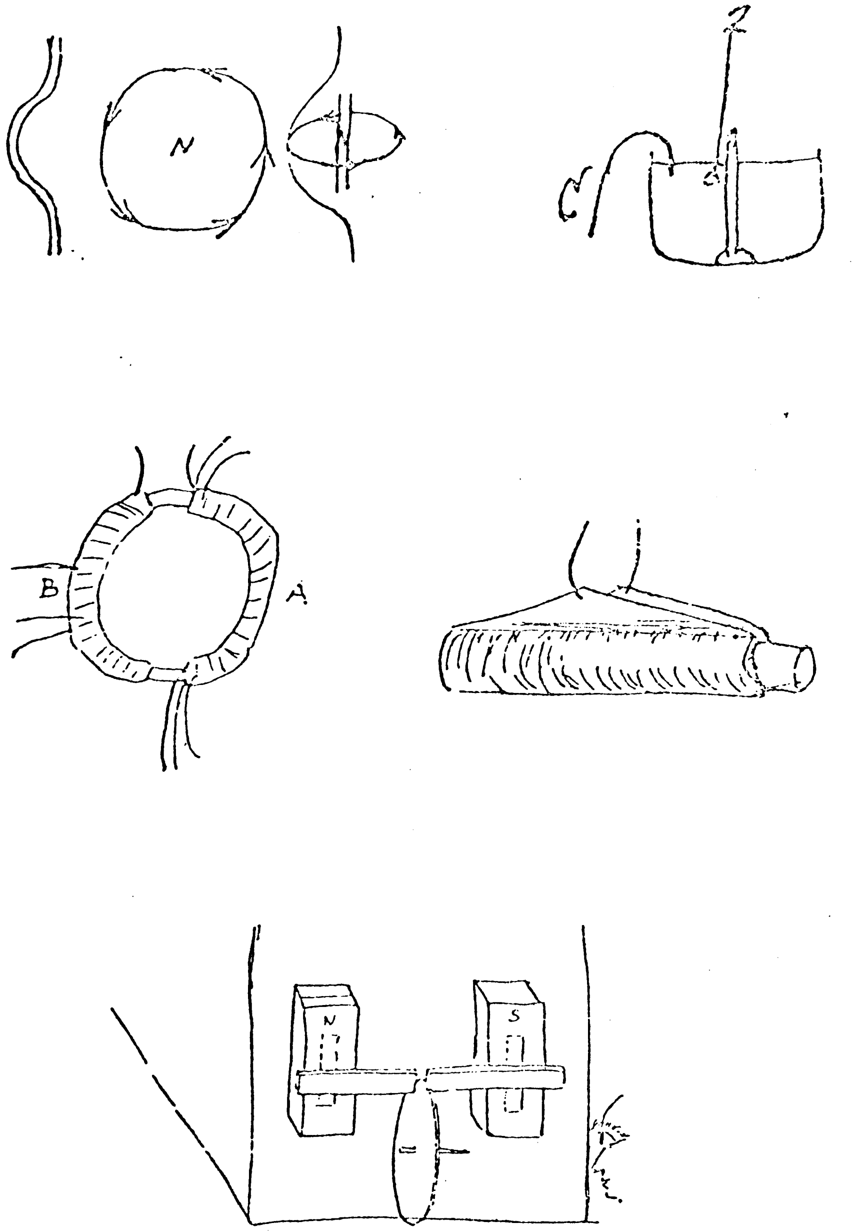

Fig. 3 : Sketches from Faraday's Diary.

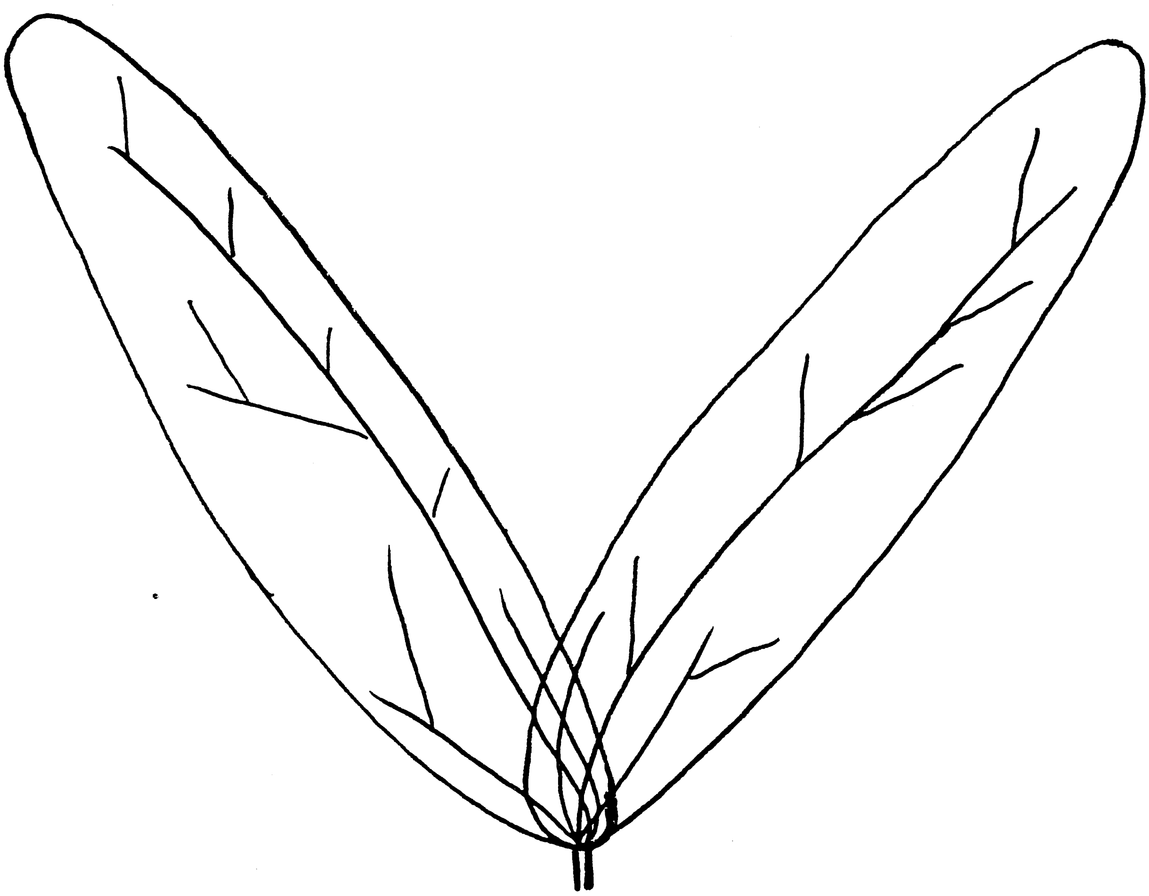

Fig. 4